<u>Theo→ then-</u> If x & y are any two vectors in a H.S.

(i) $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$

This is known as Parallelogram Law

(ii) $4(x,y) = \|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2$

This is known as Polarisation identity.

<u>Pf→</u> (i) we have→

$\|x+y\|^2 = (x+y, x+y)$

$= (x,(x+y)) + (y,(x+y))$

$= (x,x) + (x,y) + (y,x) + (y,y)$

$= \|x\|^2 + (x,y) + (y,x) + \|y\|^2 \to (1)$

Also,

$\|x-y\|^2 = (x-y, x-y)$

$= (x, x-y) - (y, x-y)$

$= (x,x) - (x,y) - (y,x) + (y,y)$

$= \|x\|^2 - (x,y) - (y,x) + \|y\|^2 \to (2)$

Adding (1) & (2), we get—

$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$

which is reqd parallelogram law.

(ii) Subtracting (2) from (1), we get—

$\|x+y\|^2 - \|x-y\|^2 = 2(x,y) + 2(y,x) \to (3)$

Replacing y by iy in (3), we get

$$\|x+iy\|^2 - \|x-iy\|^2 = 2(x, iy) + 2(iy, x)$$
$$= 2\bar{i}(x,y) + 2i(y,x)$$
$$= -2i(x,y) + 2i(y,x), \quad \because \bar{i} = -i \qquad \to (4)$$

multiplying both sides of (4) by 'i', we get —

$$i\|x+iy\|^2 - i\|x-iy\|^2 = -2i^2(x,y) + 2i^2(y,x)$$
$$= 2(x,y) - 2(y,x), \quad \because i^2 = -1 \qquad \to (5)$$

Adding (3) & (5), we get —

$$\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2 = 4(x,y)$$

which is ~~read~~ Polarization identity.

Ex:- If L is an IPS, then to show that $\sqrt{(x,x)}$ has the properties of a norm.

Soln Given that L is an IPS.

we write $\|x\| = \sqrt{(x,x)}$

Now to show that $\|x\|$ satisfies all the properties of norm i.e (i) $\|x\| \geq 0$ & $\|x\| = 0 \Leftrightarrow x = 0$

(ii) $\|\alpha x\| = |\alpha| \, \|x\|$

(iii) $\|x+y\| \leq \|x\| + \|y\|$

(i). By defn, $\|x\| = \sqrt{(x,x)} \Rightarrow \|x\|^2 = (x,x)$

Now by defn of inner product, $(x,x) \geq 0$

& $(x,x) = 0 \Leftrightarrow x = 0$

$\therefore \|x\| \geq 0$ & $\|x\| = 0 \Leftrightarrow x = 0$

(ii) we have,

$$\|\alpha x\|^2 = (\alpha x, \alpha x)$$
$$= \alpha (x, \alpha x)$$
$$= \alpha \bar{\alpha} (x, x) = |\alpha|^2 \|x\|^2$$

$$\|\alpha x\|^2 = |\alpha|^2 \|x\|^2$$

$$\Rightarrow \|\alpha x\| = |\alpha| \|x\|$$

(iii) we have,

$$\|x+y\|^2 = (x+y, x+y)$$
$$= (x, x) + (x, y) + (y, x) + (y, y)$$
$$= \|x\|^2 + (x, y) + \overline{(x, y)} + \|y\|^2$$
$$= \|x\|^2 + 2\,\mathrm{Re}(x, y) + \|y\|^2, \quad \because \alpha + \bar{\alpha} = 2\,\mathrm{Re}(\alpha)$$

$$\leq \|x\|^2 + 2|(x, y)| + \|y\|^2, \quad \because \mathrm{Re}(\alpha) \leq |\alpha|$$

$$\leq \|x\|^2 + 2\sqrt{(x, x)}\sqrt{(y, y)} + \|y\|^2$$

, using Schwarz inequality $|(x, y)| \leq \sqrt{(x, x)}\sqrt{(y, y)}$

$$= \|x\|^2 + 2\|x\|.\|y\| + \|y\|^2$$
$$= (\|x\| + \|y\|)^2$$

$$\therefore \|x+y\|^2 \leq (\|x\| + \|y\|)^2$$

$$\Rightarrow \|x+y\| \leq \|x\| + \|y\|$$

(Proved)